Classes of second order nonlinear differential equations reducible to first order ones by variation of parameters

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Abstract

The method of parameter variation for linear differential equations is extended to classes of second order nonlinear differential equations. This allows to reduce the latter to first order differential equations. Known classical equations such as the Bernoulli, Riccati and Abel equations are recovered in illustrated relevant examples.

Key words: Method of variation of parameters, nonlinear differential equations, Bernoulli equation, Riccati equation, Abel equation, Jacobi elliptic functions.

1 Introduction

Second order nonlinear differential equations are important for investigation of nonlinear phenomena in different fields of physics and mathematics. They are used to model a wide number of phenomena in plasma physics, solid state physics, optics, bio-hydrodynamics, chemical processes, nonlinear quantum mechanics, etc. See [1-5] and references therein. Formal methods of analytical integration of nonlinear differential equations thus appear of great interest in the theory of differential equations. To cite a few, one can mention various powerful methods, which are most familiarly used, such as group symmetry methods [6-7], tanh method [8-10], extended tanh method [11-13], sine-cosine method [14-15], Jacobi elliptic method [16-17], Backlund transforma-

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tions [18-19], inverse scattering method [20], pseudo spectral method [21] and F-expansion method [22-24].

Unfortunately, only a few types of nonlinear differential equations can be exactly solved, what explains the permanent need to recur to novel tools of handling differential equations. In such a direction, the method of reducing the order of ordinary differential equations (ODEs) can be of inestimable usefulness in most cases. Indeed, in general, the lowest the order of a differential equation is, the greatest is the possibility of finding suitable analytical methods for its resolution. Moreover, the accuracy of numerical treatment of a differential equation decreases as the order of the equation increases. Therefore, when a differential equation cannot be directly analytically solved, it is meaningful to transform it into an equation of lower order. The probability of solving the reduced equation is greater than the probability of solving the original equation. In practice, it is generally difficult to perform such a reduction.

This paper aims at proving that the usual method of parameter variation can be successfully used in some cases to reduce by one the order of nonlinear differential equations (NLDEs). In the sequel, we provide with some classes of second order nonlinear differential equations, transformable to first order ones using specific parameter variations. Concrete examples are exhibited as matter of illustration. Specifically, the classes of equations of the forms

$$(y')^m\, y'' + a(x)\,(y')^{m+1} = f(x, y, y'), \tag{1}$$

and

$$(y')^m\, y'' + a(y')(y')^{m+2} = f(x, y, y'), \tag{2}$$

where m is a positive integer and a is an integrable function, are considered. Note that, if the function $f(x, y, y') = g(x, y')$, i.e. the second members of equations (1) and (2) do not explicitly depend on the variable y, then we can perform the natural change of variables $z(x) = y'(x)$ to lower the order of these equations. Besides, if the function a is constant only in equation (1), (not necessarily constant in equation (2)), and if $f(x, y, y') = g(y, y')$, i.e. the equations (1) and (2) are autonomous, then we can perform the change of variables $w(y) = y'(x)$ to transform the above mentioned classes of equations into first order ones.

2 First class of reducible second order NLDEs

Let us consider nonlinear second order differential equations of the following type:

$$y'' + a(x)y' = F(x) + y'G(y)e^{-\int a(x)dx} \tag{3}$$

whose the linear part, i.e.

$$y'' + a(x)y' = 0 \tag{4}$$

yields the solution

$$y' = Ce^{-\int a(x)dx}, \tag{5}$$

where C is an arbitrary constant. Suppose that C is a differentiable function of both variables x and y expressed in the form

$$C = H(x) + K(y). \tag{6}$$

Then, (5) can be rewritten as

$$y' = [H(x) + K(y)]\, e^{-\int a(x)dx} \tag{7}$$

that we differentiate to obtain

$$y'' = e^{-\int a(x)dx}\left[H'_x + y'K'_y\right] - a(x)y', \tag{8}$$

where

$$H'_x = \frac{dH(x)}{dx} \quad \text{and} \quad K'_y = \frac{dK(y)}{dy}.$$

Substituting (7) and (8) into (3), we find

$$H'_x + y'K'_y = F(x)e^{\int a(x)dx} + y'G(y). \tag{9}$$

Clearly, (9) will take place if

$$H'_x = F(x)e^{\int a(x)dx} \quad \text{and} \quad K'_y = G(y). \tag{10}$$

We therefore state the following result:

Proposition 2.1 *The second order nonlinear differential equation (3) can be reduced to the first order differential equation (7), where the functions H and K are solutions of the first order differential equations (10), respectively.*

As a matter of clarity, let us consider the following example:

Example 2.1 Let us consider the function G in (3) in the form

$$G(y) = \frac{d}{dy}\left(\frac{b_0 + b_1 y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}\right), \tag{11}$$

where b_0, b_1, b_2, b_3 are arbitrary constants and c_0, c_1 are constants such that $(c_0, c_1) \neq (0, 0)$. Then equations (10) yield :

$$\begin{aligned} H(x) &= \int F(x) e^{\int a(x)dx} dx \\ K(y) &= A + \frac{b_0 + b_1 y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}, \end{aligned} \tag{12}$$

where A is an arbitrary constant of integration. Therefore, equation (7) is reduced to the well known Abel equation of second kind

$$y' = e^{-\int a(x)dx} \frac{b_0 + c_0(H(x) + A) + [c_1(H(x) + A) + b_1]\, y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}. \tag{13}$$

Finally, the equation (3) takes the form

$$y'' + a(x)y' = F(x) + y' \frac{d}{dy}\left(\frac{b_0 + b_1 y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}\right) e^{-\int a(x)dx} \tag{14}$$

and is integrable if the constants b_0, b_1, b_2, b_3, c_0, c_1 and the functions a and F are chosen in such a way that the Abel equation (13) be integrable.

In particular, for $c_0 = 1$, $c_1 = 0$ and $F = 0$, equation (13) leads to the separable Abel equation

$$y' = \left(A + b_0 + b_1 y + b_2 y^2 + b_3 y^3\right) e^{-\int a(x)dx} \tag{15}$$

whose the implicit solution is given by

$$\int \frac{dy}{A + b_0 + b_1 y + b_2 y^2 + b_3 y^3} = \int e^{-\int a(x)dx} dx + B, \tag{16}$$

where B is an arbitrary constant of integration.

It is worth noticing that a second kind Abel equation of the form

$$y' = \frac{f_3 y^3 + f_2 y^2 + f_1 y + f_0}{g_1 y + g_0}, \quad \text{with} \quad f_3 \neq 0, \tag{17}$$

where f_i, $(i = 0,\, 1,\, 2,\, 3)$, and g_j, $(j = 0,\, 1)$, are arbitrary functions of x, can be transformed into a canonical form. Indeed, using the variable change

$$\left\{ x = t,\ y = \frac{1 - g_0 u}{g_1 u} \right\}, \tag{18}$$

where t and $u = u(t)$ are the new independent and dependent variables, respectively, equation (17) becomes

$$u'_t = \widetilde{f}_3 u^3 + \widetilde{f}_2 u^2 + \widetilde{f}_1 u + \widetilde{f}_0. \tag{19}$$

Making use of the substitution

$$u = v - \frac{\widetilde{f}_2}{3\, \widetilde{f}_3}, \tag{20}$$

equation (19) can be put in the form

$$v'_t = h_3 v^3 + h_1 v + h_0. \tag{21}$$

Now, setting

$$v = E(t) w, \quad \text{where} \quad E(t) = e^{\int h_1(t) dt}, \tag{22}$$

brings this equation to the simpler form:

$$w'_t = \widetilde{h}_3 w^3 + \widetilde{h}_0, \tag{23}$$

which, in turn, can be reduced, with the help of the new independent variable

$$s = \int \widetilde{h}_3(t) dt, \tag{24}$$

to the usual canonical form of Abel equation of the first kind

$$w'_s = w^3(s) + k(s). \tag{25}$$

The latter is integrable by various methods known in the literature. See [26] and [29] (and references therein) for a good compilation of techniques developed to solve (25) for particular expressions of $k(s)$.

3 Second class of reducible second order NLDEs

In this section, we discuss the second relevant type of nonlinear second order differential equations compilable in the following general form:

$$y'' + a(y)(y')^2 = F(x)e^{-\int a(y)dy} + y'G(y). \tag{26}$$

We first solve the left hand side part of this equation, namely

$$y'' + a(y)(y')^2 = 0 \tag{27}$$

to get

$$y' = Ce^{-\int a(y)dy}, \tag{28}$$

where C is an arbitrary constant. As in the previous section, we assume that C is a differentiable function of both x and y as follows

$$C = H(x) + K(y). \tag{29}$$

Then, the equation (28) takes the form

$$y' = [H(x) + K(y)]\, e^{-\int a(y)dy} \tag{30}$$

that we differentiate to obtain

$$y'' = e^{-\int a(y)dy}\left[H'_x + y'K'_y\right] - a(y)(y')^2. \tag{31}$$

Substituting (30) and (31) into (26), we find

$$H'_x + y'K'_y = F(x) + y'G(y)e^{\int a(y)dy}. \tag{32}$$

Clearly, (32) will take place if

$$H'_x = F(x) \quad \text{and} \quad K'_y = G(y)e^{\int a(y)dy}. \tag{33}$$

Therefore, the following statement holds:

Proposition 3.1 *The second order nonlinear differential equation (26) can be reduced to the first order differential equation (30), where the functions H and K are solutions of the two first order differential equations (33), respectively.*

For illustration, let us consider the following example.

Example 3.1 Let $a(y) = -\frac{1}{y}$ and $G(y) = \beta\, y^n$, where n is a non zero positive integer and β is a constant. Then equation (26) becomes

$$y'' - \frac{1}{y}(y')^2 = F(x)y + \beta\, y' y^n. \tag{34}$$

Equations (33) yield

$$H(x) = \int F(x)dx + A \quad \text{and} \quad K(y) = \frac{\beta}{n}\, y^n, \tag{35}$$

where A is an arbitrary constant of integration.

Therefore, equation (30) becomes the Bernoulli equation [31]

$$y' = (H(x) + A)\, y + \frac{\beta}{n}\, y^{n+1}. \tag{36}$$

The substitution $w(x) = y^{1-n}$ transforms (36) into the linear equation

$$w'_x = -n\,(H(x) + A)\, w - \beta \tag{37}$$

whose the solution is

$$w(x) = \frac{-\beta \int e^{n\int (H(x)+A)\, dx} dx + B}{e^{n\int (H(x)+A)\, dx}}, \tag{38}$$

where B is an arbitrary constant of integration.

4 Third class of reducible second order NLDEs

The third group of second order nonlinear differential equations can be expressed as

$$(y')^m\, y'' + a(x)\,(y')^{m+1} = e^{-(m+2)\int a(x)dx} F\left(y,\, y' e^{\int a(x)dx}\right). \tag{39}$$

Remark 4.1 If we set in (3) $F(x) = C\, e^{-2\int a(x)dx}$, where C is a constant, then equation (3) appears as a particular case of equations (39) considered with $m = 0$.

The linear part of equation (39)

$$(y')^m\, y'' + a(x)\,(y')^{m+1} = 0 \tag{40}$$

can be readily solved to give

$$(y')^{m+1} = K^{m+1} e^{-(m+1)\int a(x)dx}, \tag{41}$$

where K is an arbitrary constant. Now, suppose that K is a differentiable function of the variable y. Then, the equation (41) becomes

$$(y')^{m+1} = K^{m+1}(y) e^{-(m+1)\int a(x)dx}. \tag{42}$$

Differentiate (42) to obtain

$$(y')^m\, y^{(n+2)} = K^{m+1}(y) K'_y e^{-(m+2)\int a(x)dx} - a(x)\,(y')^{m+1}. \tag{43}$$

Substituting (42) and (43) into (39), we find

$$K^{m+1}(y) K'_y = F\left(y,\, K(y)\right). \tag{44}$$

We therefore set the following result:

Proposition 4.1 *The second order nonlinear differential equation (39) can be reduced to the first order differential equation (44) using (42). Furthermore, if $K(y) = \Phi(y,\, A)$, where A is an arbitrary constant, is the general solution of (44), then the general solution of (39) is given by*

$$\int \frac{dy}{\Phi(y,\, A)} = \int e^{-\int a(x)dx} dx + B, \tag{45}$$

where B is an arbitrary constant.

The following examples are of particular significance.

Example 4.1 Suppose in (39) that F has one of the following forms

$i)\; F(u,\, v) = u^{m+1}\, f\left(\dfrac{au + bv + c}{\alpha u + \beta v + \gamma}\right)$

or

$$ii) F(u,\, v) = v^{m+1}\, f\left(\frac{au + bv + c}{\alpha u + \beta v + \gamma}\right), \tag{46}$$

where α, β, γ, a, b and c are constants such that the function f is well defined. Then, using the method developed in [26] and [28], the corresponding first order differential equation (44) can be transformed into a homogeneous equation which, in turn, can be reduced to a separable equation. As a matter of illustration, let us perform such a transformation in the case where F is of the form ii).

(1) If $\Delta = a\beta - b\alpha \neq 0$, the transformations

$$y = u + \frac{b\gamma - c\beta}{\Delta} \quad \textit{and} \quad K(y) = v(u) + \frac{c\alpha - a\gamma}{\Delta} \tag{47}$$

lead to the equation

$$v'_u = f\left(\frac{au + bv}{\alpha u + \beta v}\right). \tag{48}$$

Dividing both the numerator and the denominator of the fraction on the right-hand side by u, we obtain the homogeneous equation

$$v'_u = f\left(\frac{a + b\frac{v}{u}}{\alpha + \beta\frac{v}{u}}\right) \equiv \widetilde{f}\left(\frac{v}{u}\right), \tag{49}$$

for which the substitution $w(u) = \frac{v}{u}$ gives the separable equation

$$u\, w'_u = \widetilde{f}(w) - w. \tag{50}$$

(2) For $\Delta = 0$ and $b \neq 0$, the substitution $v(y) = ay + bK(y) + c$ engenders the separable equation

$$v'_y = a + bf\left(\frac{bv}{\beta v + b\gamma - c\beta}\right). \tag{51}$$

(3) For $\Delta = 0$ and $\beta \neq 0$, from the substitution $v(y) = \alpha y + \beta K(y) + \gamma$, we deduce the separable equation

$$v'_y = \alpha + \beta f\left(\frac{bv + c\beta - b\gamma}{\beta v}\right). \tag{52}$$

Setting $m = 0$, $a(x) = \frac{1}{x}$ and $F(u,\, v) = v\, \left[\left(\frac{v}{u}\right)^2 + 2\, \left(\frac{v}{u}\right)\right]$, the equation (39) becomes

$$y'' + \frac{1}{x}\, y' = x\, y'\, \left(\frac{y'}{y}\right)^2 + 2\, \frac{y'^2}{y}. \tag{53}$$

Perform in (53) the substitution (42) which takes the form

$$y' = \frac{K(y)}{x}. \tag{54}$$

By (44), the function K satisfies the homogeneous equation

$$K'_y = \left(\left(\frac{K}{y}\right)^2 + 2\,\frac{K}{y}\right) \tag{55}$$

which gives the solution

$$K(y) = \frac{y^2}{A - y}, \tag{56}$$

where A is an arbitrary constant of integration. Hence, using the integral (45), we obtain, for equation (53), the implicit solution

$$-\frac{A}{y} - \ln(y) = \ln(x) + B, \tag{57}$$

where B is an integration constant.

Suppose that F, defined in (39), has the following general form

$$F(u,\, v) = \frac{\sum_{\nu=1}^{p} h_\nu(u)\, F_\nu(v)}{\sum_{\eta=1}^{q} k_\eta(u)\, G_\eta(v)}, \tag{58}$$

where p, q are positive integers, h_ν, k_η, F_ν and G_η are arbitrary functions such that F be well defined.

Example 4.2 If F in (58) is of the form

$$F(u,\, v) = h_1(u)\, R\left(v,\, \sqrt{P(v)}\right), \tag{59}$$

where R is a rational function of two variables and the function P under the radical is a polynomial of degree three or four, then the corresponding reduced equation (44) readily turns to be an elliptic integral which can be merely integrated by the method presented in [32].

Example 4.3 If F in (58) has the form

$$F(u,\, v) = v^{m+1}(h_1(u)\, v + h_2(u)\, v^n), \quad n \in \mathbb{N} \setminus \{0, 1\}, \tag{60}$$

then the corresponding reduced equation (44) leads to a Bernoulli equation. With $m = 0$, $a(x) = 2/x$ and $F(u,\, v) = v\ (v + v^3)$, the equation (39) becomes

$$y'' + \frac{2}{x}\, y' = (y')^2 + (x\, y')^4 . \tag{61}$$

Perform in (61) the substitution (42) which takes the form

$$y' = \frac{K(y)}{x^2}. \tag{62}$$

By (44), the function K satisfies the Bernoulli equation

$$K'_y = K + K^3 \tag{63}$$

yielding the solution

$$K(y) = \pm \left(A\, e^{-2\, y} - 1\right)^{-\frac{1}{2}}, \tag{64}$$

where A is an arbitrary constant of integration.

Hence, by using the integral (58), the equation (61) provides an implicit solution

$$\arctan\left(\sqrt{A\, e^{-2\, y} - 1}\right) - \sqrt{A\, e^{-2\, y} - 1} = \frac{1}{x} + B, \tag{65}$$

where B is an integration constant.

Example 4.4 If F in (58) has the form

$$F(u,\, v) = v^{m+1}(h_1(u) + h_2(u)\, v + h_3(u)\, v^2), \tag{66}$$

then the corresponding reduced equation (44) leads to a Riccati equation. An important number of integrable Riccati equations is recorded in [27-30]. Recall that, if $y_0 = y_0(x)$ is a given particular solution of the Riccati equation

$$y' = f(x)y^2 + g(x)y + h(x), \tag{67}$$

then, the general solution can be written as:

$$y(x) = y_0(x) + \Phi(x) \left[C - \int f(x)\Phi(x)dx\right]^{-1}, \tag{68}$$

where

$$\Phi(x) = \exp\left\{ \int \left[2f(x)y_0(x) + g(x)\right] dx \right\}; \tag{69}$$

C is an arbitrary constant. To the particular solution $y_0(x)$ there corresponds $C = \infty$.

Example 4.5 Let F in (58) be of the form

$$F(u,\, v) = v^{m+1} \frac{h_1(u) + h_2(u)\, v + h_3(u)\, v^2 + h_4(u)\, v^3}{k_1(u) + k_2(u)\, v}. \tag{70}$$

Then the corresponding reduced equation (44) leads to an Abel equation.

As a matter of fact, let's briefly present how to transform, into its canonical form, the Abel equations of the second kind:

$$y' = \frac{f_2\, y^2 + f_1\, y + f_0}{g_1\, y + g_0}, \quad \text{with} \quad f_2 \neq 0, \tag{71}$$

where f_i, $i = 0,\, 1,\, 2$, and g_j, $j = 0,\, 1$, are arbitrary functions of x.

Using the variable change

$$y = \frac{z - g_0}{g_1}, \tag{72}$$

equation (71) takes the form

$$z\, z' = \widetilde{f}_2\, z^2 + \widetilde{f}_1\, z + \widetilde{f}_0. \tag{73}$$

Now, the substitution

$$z = E(x)\, w \quad \text{where} \quad E(x) = e^{\int \widetilde{f}_2(x) dx} \tag{74}$$

brings this equation to the simpler form :

$$w\, w'_x = \widetilde{h}_1\, w + \widetilde{h}_0, \tag{75}$$

which, in turn, can be reduced, by the introduction of the new independent variable

$$s = \int \widetilde{h}_1(x) dx, \tag{76}$$

into the canonical form of the second kind Abel equation

$$w(s)\, w'_s = w(s) + k(s). \tag{77}$$

A good compilation of integrable Abel equations of the form (77) can be found in [27-30].

Considering $m = 0$, $a(x) = -\frac{1}{x}$ and $F(u,\, v) = v + 2\, u$, the equation (39) can be reduced to the linear equation

$$y'' - \left(\frac{1}{x} + x\right) y' - 2\, x^2\, y = 0. \tag{78}$$

Perform in (78) the substitution (42) which takes now the form

$$y' = x\, K(y). \tag{79}$$

By (44), the function K verifies the Abel equation of second kind in its canonical form

$$K\, K'_y = K + 2\, y. \tag{80}$$

Two particular solutions of equation (80) are given by

$$K_1(y) = 2\, y \quad \text{and} \quad K_2(y) = -y \tag{81}$$

from which can be deduced a general solution satisfying the algebraic equation

$$\left(K(y) - 2\, y\right)^2\, \left(K(y) + y\right) = A, \tag{82}$$

where A is an arbitrary constant. A real solution of the latter can be computed to yield

$$\begin{aligned} K(y) &= \frac{1}{2}\sqrt[3]{-8\, y^3 + 4\, A + 4\sqrt{-4\, A\, y^3 + A^2}} \\ &\quad + \frac{2\, y^2}{\sqrt[3]{-8\, y^3 + 4\, A + 4\sqrt{-4\, A\, y^3 + A^2}}} + y \\ &\equiv \Phi(y,\, A). \end{aligned}$$

Hence, by using the integral (58), we obtain an implicit solution of the equation (78) as

$$\int [\Phi(y,\, A)]^{-1} = \frac{1}{2}\, x^2 + B, \tag{83}$$

where B is a constant of integration.

5 Fourth class of reducible second order NLDEs

Let us now investigate second order nonlinear differential equations of the following form:

$$(y')^m\, y'' + a(y)\, (y')^{m+2} = e^{-(m+1)\int a(y)dy} F\left(x,\, y' e^{\int a(y)dy}\right), \tag{84}$$

Remark 5.1 It is immediate to note that, if we set in (26) $G(y) = C$, where C is a constant, then equation (26) appears as a particular case of equation (84) with $m = 0$.

We first solve, as in the previous sections, the left hand side part of equation (84), namely

$$(y')^m\, y'' + a(y)\, (y')^{m+2} = 0 \tag{85}$$

from which follows

$$(y')^{m+1} = K^{m+1} e^{-(m+1)\int a(y)dy}, \tag{86}$$

where K is an arbitrary constant. Suppose that K is a differentiable function of the variable x. Then, equation (86) becomes

$$(y')^{m+1} = K^{m+1}(x) e^{-(m+1)\int a(y)dy}. \tag{87}$$

Differentiate (87) to obtain

$$(y')^m\, y'' = K^m(x) K'_x e^{-(m+1)\int a(y)dy} - a(y)\, (y')^{m+2}\,. \tag{88}$$

Substituting (87) and (88) into (84) we find

$$K^m(x) K'_x = F\left(x,\, K(x)\right). \tag{89}$$

We therefore arrive at the following result:

Proposition 5.1 *The second order nonlinear differential equation (84) can be reduced to the first order differential equation (89) using (87). Furthermore, if* $K(x) = \Phi(x,\, A)$, *where* A *is an arbitrary constant, is the general solution of (89), then the general solution of (84) is given by*

$$\int e^{\int a(y)dy} dy = \int \Phi(x,\, A)dx + B, \tag{90}$$

where B *is an arbitrary constant.*

Remark 5.2

(1) The equations (39) and (84) have been investigated by Jovan in [25] *for* $m = 0$. Furthermore, this author has also examined some examples of integrable equations corresponding to particular cases where F is of the form (58) with $q = 1$, $k_1(u) = G_1(v) = 1$ and $F_\nu(v) = v^{\alpha_\nu}$, with $\alpha_\nu \in \mathbb{R}$, $\nu = 1,\, \ldots,\, p$.

(2) The equations (39) and (84), considered with $m \neq 0$, are equivalent to the equations of the same type with $m = 0$.

The examples given in the previous section can be also considered in the framework of the Proposition 5.1. Namely, one can easily modify those examples to obtain equations of the form (84) which can be integrated by quadrature in simpler situations or reduced at least to first order equations in more cumbersome cases.

As a matter of illustration, let us consider the following examples in which the function F is of the form (58) and $m = 0$.

Example 5.1 Let $a(y) = -1/y$ and $F(u,\, v) = \sqrt{(1 - v^2)\,(1 - k^2 v^2)}$ where $k \in \mathbb{R}^\star$. Equation (84) becomes

$$y'' - \frac{1}{y}\,(y')^2 = y\, \left[\left(1 - \left(\frac{y'}{y}\right)^2\right) \left(1 - k^2\, \left(\frac{y'}{y}\right)^2\right) \right]^{\frac{1}{2}}. \tag{91}$$

Perform in (91) the substitution (87) which takes the form

$$y' = K(x)\, y. \tag{92}$$

By (89), the function K satisfies the elliptic integral

$$\int \frac{dK}{\sqrt{(1 - K^2)\,(1 - k^2\, K^2)}} = x + A, \tag{93}$$

where A is an arbitrary constant of integration. Equation (93) is readily solved to give

$$K(x) = \mathrm{sn}(x + A,\, k), \tag{94}$$

where sn is the first Jacobi elliptic function. Hence, by using the integral (90), the solution of the equation (91) is obtained in the form

$$y(x) = B\, \left[\mathrm{dn}(x + A,\, k) + k\, \mathrm{cn}(x + A,\, k)\right]^{-\frac{1}{k}}, \tag{95}$$

where B is an integration constant; cn and dn stand for the second and third Jacobi elliptic functions, respectively.

Example 5.2 Let $a(y) = -\frac{1}{y}$ and $F(u,\, v) = \frac{b}{u^2} + a\, v^2,\ a,\, b \in \mathbb{R}^\star$. Equation (84) becomes

$$y'' - \frac{1}{y}\,(y')^2 = y\, \left[\frac{b}{x^2} + a\, \left(\frac{y'}{y}\right)^2\right]. \tag{96}$$

Perform in (96) the substitution (87) which takes the form

$$y' = K(x)\, y. \tag{97}$$

By (89), the function K verifies the Riccati equation

$$K'_x = \frac{b}{x^2} + a\, K^2 \tag{98}$$

which can be solved to yield

$$K(x) = \frac{\lambda}{x} - x^{2\,a\,\lambda}\, \left(\frac{a\, x^{2\,a\,\lambda+1}}{2\,a\,\lambda + 1} + A\right)^{-1}, \tag{99}$$

where A is an arbitrary constant of integration and λ is a solution of the quadratic equation $a\,\lambda^2 + \lambda + b = 0$. Hence, by using the integral (90), we obtain, for equation (96), the solution

$$y(x) = \frac{B\, x^\lambda}{(a\, x\, e^{2\,a\,\lambda\, \ln x} + 2\,a\,A\,\lambda + A)^{\frac{1}{a}}}, \tag{100}$$

where B is a constant of integration.

6 Concluding remarks

We have investigated, in this paper, four different classes of second order nonlinear differential equations which have been reduced to first order ones, using suitable parameter variations. Fortunately, the resulting first order differential equations are, in most cases, transformable to well known integrable or solvable classical differential equations whose the solutions can be worked out by various methods disseminated in the standard specialized text books. Finally, it appears possible to extend the parameter variation methods developed in this work to classes of higher order nonlinear differential equations with a view to their order reduction. Such an investigation will be in the core of forthcoming work.

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